# Calculation of $C_n^2$ for visible light and sound from CSAT3 sonic anemometer measurements

Steve Oncley and Tom Horst, NCAR/ATD

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## 1 Introduction

For DASH, ATD is planning to make measurements which can be used to obtain $C_n^2$ for visible light to be compared to scintillometer measurements that already have been taken. We will deploy a sonic anemometer to measure acoustic temperature $T_{sv}$ from a measurement of the speed of sound and a krypton hygrometer to measure specific humidity. (Actually, this hygrometer has a temperature contamination which would have to be dealt with if needed.)

This document presents the various steps to calculate $C_n^2$ for visible light and the structure parameter for sound $C_{na}^2$ from measurements of $C_{T_{sv}}^2$. Note that the index of refraction for electromagnetic waves, and thus $C_n^2$, is wavelength dependent, and is characterized by different weightings of the temperature and humidity structure parameters. Thus, the final equations below are not appropriate for light in the infrared or ultraviolet. (This should be acceptable for DASH since we are comparing our measurements with scintillometers working at visible wavelengths.)

## 2 Structure function parameters

A structure function of a quantity $a$ is defined as

$$C_a^2(\Delta x) = \frac{< a(x)^2 - a(x + \Delta x)^2 >}{\Delta x^{2/3}} \qquad (1)$$

where $<>$ denote averaging over the space $x$. For scales within the inertial subrange, $C_a^2$ should be independent of $\Delta x$. Using Taylor's hypothesis, Eq. 1 may be transformed to an average over time $t$.

$$C_a^2 = \frac{\overline{a(t)^2 - a(t+\Delta t)^2}}{(\overline{U}\Delta t)^{2/3}} \tag{2}$$

where variables are now averaged over time (overbar) and $U$ is the longitudinal wind component.

Muschinski et al. (2001) show that $C_a^2$ may be calculated from the power spectrum $S_a$ of $a$ in the inertial subrange using

$$C_a^2 = 13.67 S_a(f) f^{5/3} U^{-2/3} \tag{3}$$

where $f$ is frequency and 13.67 is a constant containing a factor 0.249 and $(2\pi)^{2/3}$.

Thus, the structure parameter is readily computed from the amplitude of the power spectra of a quantity in the inertial subrange (the high-frequency part of the spectrum where the power decreases as $f^{-5/3}$).

# 3 $C_n^2$ as a function of $C_T^2$ and $C_Q^2$

We now follow the derivation of Hill et al. (1980), but only considering the propagation of visible light and assuming that pressure fluctuations have a much smaller effect on $C_n^2$ than temperature or humidity fluctuations. Then,

$$N = N_d + N_w \tag{4}$$

where $N$ is the refractivity of the air with contributions $N_d$ from dry air and $N_w$ from water vapor. Note that refractivity is defined as

$$N = (n-1) \times 10^6 \tag{5}$$

where $n$ is the actual index of refraction (a value close to unity).

Hill et al. give the following expressions for $N_d$ and $N_w$ as functions of temperature $T$ (degrees K), water vapor density $Q$ (molecules/m$^3$), pressure $P$ (Torr), and wavelength $\lambda$ ($\mu$m).

$$N_d = \frac{0.3789P}{T} N_0[1 + (5.337 - 0.0157T) \times 10^{-6} P] \tag{6}$$

where

$$N_0 = 64.328 + 29498.1/(146 - \lambda^{-2}) + 255.4/(41 - \lambda^{-2}) \tag{7}$$

and

$$N_w = -1.765 \times 10^{-18}(1 - 0.0109\lambda^{-2})Q. \tag{8}$$

Hill et al. continue by neglecting the (small) wavelength dependence and second-order $T$ and $P$ dependence in Eq. 6 to produce:

$$N_d = 0.3789 N_0 \frac{P}{T} \tag{9}$$

and

$$N_w = -1.765 \times 10^{-18} Q. \tag{10}$$

For easier use, we convert the units of $P$ to Pascals and $Q$ to kg/kg, which changes the above two equations to

$$N_d = 0.00284 N_0 \frac{P}{T} \tag{11}$$

and

$$N_w = -59\rho Q \tag{12}$$

where $\rho$ is the density of air.

In calculations below, we let $T$=273.15 K, $Q$=0.004 kg/kg, and $P$=840 mb = 84000 Pa.

Differentiating,

$$dN = -N_d \frac{dT}{T} + N_w \frac{dQ}{Q}. \tag{13}$$

Referencing each variable to a mean (overbar) and fluctuation ('), Eq. 9 becomes

$$N' = -\overline{N_d} \frac{T'}{\overline{T}} + \overline{N_w} \frac{Q'}{\overline{Q}}. \tag{14}$$

Using Eq. 5,

$$n' = A_T \frac{T'}{\overline{T}} + A_Q \frac{Q'}{\overline{Q}}. \tag{15}$$

where

$$A_T = -\overline{N_d} \times 10^{-6} \tag{16}$$

and

$$A_Q = \overline{N_w} \times 10^{-6}. \tag{17}$$

Note that de Bruin et al. (1995) use for light at 0.94 $\mu$m

$$A_T = -0.78 \frac{P}{T} \times 10^{-6} \tag{18}$$

and

$$A_Q = -59Q \times 10^{-6}. \tag{19}$$

(Andreas (1987) found coefficients of -0.79 and -56.4 in the above 2 equations for light at 0.55 $\mu$m, with $Q$ in kg/m$^3$.) For $\lambda = 0.94$ $\mu$m, $N_O$=274, and $\rho$=1 kg/m$^3$, the coefficients in Eq. 11-12 and 18-19 agree.

From Eq. 15, an equation for $C_n^2$ can be presented

$$C_n^2 = C_T^2 \frac{A_T^2}{\overline{T}^2} + C_Q^2 \frac{A_Q^2}{\overline{Q}^2} + 2C_{TQ} \frac{A_T A_Q}{\overline{TQ}}. \tag{20}$$

To estimate the relative importance of the $C_T$ and $C_Q$, assume

$$C_{TQ} = r_{TQ} C_T C_Q \tag{21}$$

and let $r_{TQ} = \pm 1$ (Wesley, 1976). Then, using Eq. 18-19, Eq. 20 simplifies to:

$$\begin{aligned} C_n &= C_T \frac{A_T}{\overline{T}} \left( 1 + \frac{A_Q}{A_T} \frac{C_Q}{C_T} \frac{\overline{T}}{\overline{Q}} \right) \\ &= C_T \frac{A_T}{\overline{T}} \left( 1 + \frac{-59Q}{-0.78P/T} \frac{\overline{w'Q'}/\rho L}{\overline{w'T'}/\rho C_{pair}} \frac{\overline{T}}{\overline{Q}} \right) \\ &= C_T \frac{A_T}{\overline{T}} \left( 1 + \frac{-59}{-0.78} \frac{C_{pair}}{L\beta} \frac{\overline{T}^2}{P} \right) \\ &= C_T \frac{A_T}{\overline{T}} \left( 1 + \frac{0.03}{\beta} \right) \end{aligned} \tag{22}$$

where $\beta$ is the Bowen ratio $\overline{w'T'}/\overline{w'Q'}$. If a 10% error is acceptable, the right-most quantity in () may be ignored for $\beta$ >0.3. NCAR/ATD has flux data in January from Table Mountain, just north of Boulder, which should be similar to that expected for the BAO. These data indicate that $|\beta > 0.3|$ about 90% of the time during the day.

# 4 Acoustic temperature

Because NCAR/ATD does not have temperature probes with adequate frequency response to measure $T'$, we plan to use the acoustic virtual temperature $T_{sv}$ from our sonic anemometers

$$T_{sv} = T(1 + fQ) = T + fTQ \tag{23}$$

where $f$ is $0.51 \times 10^{-3}$. (Note that this is slightly different than actual virtual temperature for which $f = 0.61 \times 10^{-3}$.)

NCAR/ATD plan to measure humidity fluctuations with our krypton hygrometers (which also, unfortunately, have a slight temperature dependence), but these sensors must be physically separated (to avoid impacting the air flow through the sonic anemometer). Thus, at the smallest spatial scales our measurements of $T_{sv}$ and $Q$ will be decorrelated, so it will be impossible to use the above equation to calculate $T$ itself. For this reason, we must now derive how $C_n^2$ can be calculated from $C_{T_{sv}}^2$.

Differentiating Eq. 22

$$T'_{sv} = (1 + f\overline{Q})T' + f\overline{T}Q'. \tag{24}$$

Thus

$$\begin{aligned} C_{T_{sv}} &= C_T \left(1 + f\overline{Q} + f\overline{T}\frac{C_{pair}}{\beta L}\right) \\ &= C_T \left(1 + 0.002 + \frac{0.06}{\beta}\right). \end{aligned} \tag{25}$$

Combined with Eq. 22

$$C_n \approx C_{T_{sv}} \frac{A_T}{\overline{T}} \left(1 - \frac{0.03}{\beta}\right) \tag{26}$$

which is gives the same error as a function of Bowen ratio as in Eq. 22, but of the opposite sign. Thus, for small Bowen ratio, $\beta$, it is reasonable just to use sonic temperature.

# 5 Formulation for sound propagation

For the refractive index of acoustic waves $C_{na}$, again for $r_{TQ} = \pm 1$, we have from Wesley

$$C_{na} = \left(\frac{C_T}{2\overline{T}}\right)\left(1 + \frac{0.06}{\beta}\right). \tag{27}$$

In other words, the effect of humidity on sound scattering is exactly the same as for acoustic temperature, which is not surprising. Thus:

$$C_{na} \approx \frac{C_{T_{sv}}}{2\overline{T}} \tag{28}$$

which obviously will make the comparison between the sonic anemometer and sodar measurements trivial.

# 6 Frequency–dependent corrections to the inertial–subrange power spectra

The preceding calculations rely on measurements of turbulence in the inertial subrange, which occurs at spatial scales that are on the order of the path length of the sonic anemometer and at time scales on the order of both the sampling rate of the data and the timing sequence of sound pulses among the three sonic measurement axes. Consequently the power spectra are corrected for aliasing, sonic path averaging, and the influence of pulse sequence delays on sonic response. Prior to these corrections, an independent correction has also been applied to the temperature spectra for white noise associated with the resolution threshold of the CSAT3 sonic anemometer. In the following, the corrections are discussed individually and then combined into an overall formula for correction of the inertial–subrange spectra.

## 6.1 Aliasing

As a consequence of digital sampling of a physical process at a rate $f_s$, only frequencies within the range $0 \leq f \leq f_N$ can be resolved from the measurements, where the Nyquist frequency $f_N = f_s/2$. However, energy in the frequency range $f > f_N$ is not lost, but is 'folded' back to the observed range of frequencies, $0 \le f \le f_N$. Thus the aliased (observed) spectrum is

$$S^a(f) = S(f) + \sum_{m=1}^{\infty} H(2mf_N - f)S(2mf_N - f) + \sum_{m=1}^{\infty} H(2mf_N + f)S(2mf_N + f) \quad , \tag{29}$$

where $S(f)$ is the true spectrum and $H(f)$ is the frequency–dependent attenuation caused by processes such as sonic path averaging and pulse sequence delay. If $f$ is in the inertial subrange, then

$$S(f) = S^a(f) \Big/ \sum_{m=-\infty}^{\infty} H(f_m)(f_m/f)^{-5/3} \tag{30}$$

where $f_m \equiv |f + 2mf_N/|^{-5/3}$

## 6.2 White noise removal

Since the preceding correction for aliasing assumes that the spectrum has an inertial–subrange decay of $f^{-5/3}$, any noise in the observed spectrum must be removed prior to applying the multiplicative correction. The manual for the CSAT3 specifies that instantaneous temperature measurements made with a constant input have a standard deviation $\sigma_n$ that is independent of the sample rate and equal to 0.002°C. Since inspection of the observed temperature power spectra suggest that the actual noise level may be up to twice that value, we have subtracted a constant value from the spectra equal to $4\sigma_n^2/f_N$. With $f_N = 15$ sec$^{-1}$ and $U = 3$ m/s, Eqs. 3 and 26 imply that the noise level for $C_n^2$ at the Nyquist frequency is $5 \times 10^{-16}$ m$^{-2/3}$.

## 6.3 Sonic pulse sequence delays

Larsen et al. (1993) examine attenuation and cross–contamination of temperature and velocity measurements that occur when sonic anemometer acoustic pulses in opposite directions along a single measurement path are emitted sequentially rather than simultaneously. Nielsen and Larsen (2002) extend this analysis to the R3 Solent sonic anemometer, which has a path geometry similar to the CSAT3 and, like the CSAT3, outputs a temperature that is an average over the three measurement paths and is corrected for sound path curvature, i.e. cross–contamination by the velocity component normal to the measurement path.

For a single measurement path, Larsen et al. (1993) show that

$$T'_S = \frac{1}{2}\left[\, T'_{sv}(t_1) + T'_{sv}(t_2)\,\right] + \frac{a}{2}\left[\, w'(t_1) - w'(t_2)\,\right] - \frac{a}{2}\frac{U}{\bar{c}}\left[\, u'(t_1) + u'(t_2)\,\right] \quad (31)$$

$$w'_S = \frac{1}{2}\left[\, w'(t_1) + w'(t_2)\,\right] + \frac{1}{2a}\left[\, T'_{sv}(t_1) - T'_{sv}(t_2)\,\right] - \frac{1}{2}\frac{U}{\bar{c}}\left[\, u'(t_1) - u'(t_2)\,\right] \quad (32)$$

where $T_S$ is the temperature and $w_S$ the wind measured and output by the single–path sonic anemometer, $T_{sv}$ is the true acoustic virtual temperature, $u$ and $w$ are the wind components normal and parallel to the sonic measurement path, $t_1$ and $t_2$ refer to the sequential times at which the two acoustic pulses are emitted in opposite directions along the sonic path, $c$ is the speed of sound, and

$$a \equiv \frac{2\bar{T}_{sv}}{\bar{c}} \quad (33)$$

is about 1.8. If the two pulses were emitted simultaneously, then the second terms in both equations and the third term in Eq. 32 equal zero and the measured temperature is in error by only the third term in Eq. 31, the sound path curvature error mentioned previously. The CSAT3 sonic corrects the measured temperature for this error, but with the assumption that the pulses are emitted simultaneously. Measurement of the crosswind component for each path necessarily entails measurements by two more (non–parallel) paths, and therefore this correction is only partially effective since it also uses data obtained at times $t_3$ through $t_6$. However, we note that the ratio of the third term to the second term in Eqs. 31–32 is on the order of the Mach number $U/\bar{c}$ and therefore neglect the third terms in the following.

The CSAT3 sonic anemometer measures three wind components and virtual temperature using three non–orthogonal paths (a,b,c), each at an angle $\phi = \pi/3$ with respect to the horizontal and intersecting the horizontal plane at intervals of $2\pi/3$. The sonic temperature output by the CSAT3 is the average of the measurements for all three paths. Thus

$$\begin{aligned} T'_S = {} & \frac{1}{6}\sum_{j=1}^{6} T'_{sv}(t_j) \\ & + \frac{a}{6}\left[\, w'_a(t_1) - w'_a(t_2) + w'_b(t_3) - w'_b(t_4) + w'_c(t_5) - w'_c(t_6)\,\right] \quad . \end{aligned} \quad (34)$$

The time interval between sequential pulses, e.g. $t_2 - t_1 = 2\tau$, where the time required for a single set of 6 pulses, $12\tau = 13.4$ msec for a CSAT3 sonic collecting 30 samples per second. (Fig. 2 of Larsen, et al., 1993, incorrectly associates $\tau$ with the time between pulses; Nielsen and Larsen, 2002, correct this error.) Following Larsen et al. (1993), the Fourier transform of Eq. (34) is

$$\begin{aligned} dZ_{T_S} = \frac{1}{3}\, dZ_{T_{sv}} \cos\omega\tau \,(\, 1 + 2\cos 4\omega\tau) \\ - \frac{ia}{3}\, \sin\omega\tau \left[ dZ_{w_a} \mathrm{e}^{-i4\omega\tau} + dZ_{w_b} + dZ_{w_c} \mathrm{e}^{i4\omega\tau} \right] \qquad , \end{aligned} \tag{35}$$

where $\omega = 2\pi f$, and the power spectrum of $T_S$ is

$$\begin{aligned} S_{T_S} = \frac{1}{9} S_{T_{sv}} \cos^2\omega\tau \,(\, 1 + 2\cos 4\omega\tau)^2 + \frac{a^2}{9} \sin^2\omega\tau \,[\, S_{w_a} + S_{w_b} + S_{w_c} \\ + 2\,(Co_{w_a w_b} + Co_{w_b w_c}) \cos 4\omega\tau + 2\, Co_{w_c w_a} \cos 8\omega\tau \\ - 2\,(Q_{w_a w_b} + Q_{w_b w_c}) \sin 4\omega\tau + 2\, Q_{w_c w_a} \sin 8\omega\tau \,] \\ + \frac{a}{9} \sin 2\omega\tau \,(\, 1 + 2\cos 4\omega\tau) \,[\, (Co_{w_c T_{sv}} - Co_{w_a T_{sv}}) \sin 4\omega\tau \\ - (Q_{w_a T_{sv}} + Q_{w_c T_{sv}}) \cos 4\omega\tau - Q_{w_b T_{sv}} \,] \qquad . \end{aligned} \tag{36}$$

Thus the spectrum of sonic temperature is attenuated by the factor

$$H^{\tau}_{T_{sv}}(\omega) = \frac{1}{9} \cos^2\omega\tau (\, 1 + 2\cos 4\omega\tau)^2 \tag{37}$$

and contaminated by the spectra of the velocity components parallel to the sonic paths, as well as by cospectra $Co$ and quadrature spectra $Q$ between those velocities and between those velocities and sonic temperature. The (aliased) contamination, like the measurement noise, must be subtracted from the measured sonic temperature spectra prior to correction for path averaging and aliasing. The net spectral correction will be discussed in Section 6.5.

Equations for the measured values of the velocity spectra and cross spectra, as well as the velocity–temperature cross spectra are found by the same method used to obtain Eq. 36, thus requiring the simultaneous solution of multiple equations in multiple unknowns to obtain estimates of the true spectral values. Here we simplify that calculation by assuming that the contamination terms in all such equations are small and we only retain the leading attenuation term in all the equations required to provide the contamination terms in Eq. 36. These are,

$$\tilde{S}_w \simeq S_w \cos^2 \omega\tau \tag{38}$$

$$\tilde{Co}_{w_a w_b} \simeq Co_{w_a w_b} \cos^2 \omega\tau \tag{39}$$

$$\tilde{Q}_{w_a w_b} \simeq 0 \tag{40}$$

$$\tilde{Co}_{w_a T_{sv}} \simeq Co_{w_a T_{sv}} \tag{41}$$

$$\tilde{Q}_{w_a T_{sv}} \simeq Q_{w_a T_{sv}} \cos 2\omega\tau \tag{42}$$

where, for example, $\tilde{S}_w$ is the measured value of $S_w$.

## 6.4 Sonic path averaging

The response of the anemometer is also reduced at high wavenumbers because $T_{sv}$ and $w$ are calculated as averages over the acoustic path(s). Furthermore, the CSAT3 sonic anemometer combines the data from 3 such paths for measurement of temperature. Kaimal et al. (1968) present a detailed derivation of the transfer functions, the ratio of the measured to the true one–dimensional power spectra, for the three wind components measured by a Kaijo Denki sonic anemometer. It is straightforward to extend this analysis to CSAT3 measurements of wind and temperature.

The sonic virtual temperature, averaged over a single sonic path of length $p$, is

$$\tilde{T}_{sv}(\mathbf{x}_o, \mathbf{p}) = \frac{1}{p} \int_{-p/2}^{p/2} T_{sv}(\mathbf{x}_o + \mathbf{s})\, ds \qquad . \tag{43}$$

Here $T_{sv}(\mathbf{x})$ is the sonic virtual temperature at point $\mathbf{x}$, and $\mathbf{x}_o$ is the center point of the sonic path. By representing the temperature field in terms of its Fourier components, it follows that the transfer function for the one–dimensional, streamwise power spectrum is

$$H_{T_{sv}}(k_1, \mathbf{p}) = \frac{\iint_{-\infty}^{\infty} \operatorname{sinc}^2(\mathbf{k}\cdot\mathbf{p}/2)\, \Phi_{T_{sv}}(\mathbf{k})\, dk_2\, dk_3}{\iint_{-\infty}^{\infty} \Phi_{T_{sv}}(\mathbf{k})\, dk_2\, dk_3} \qquad , \tag{44}$$

where sinc $\equiv \sin(x)/x$, $\mathbf{k} = \sum_{j=1}^{3} \hat{\mathbf{i}}_j k_j$ is wave number with components $k_1$ in the streamwise direction and $k_3$ in the vertical direction, and $\Phi_{T_{sv}}(\mathbf{k})$ is the spectral density for sonic temperature.

The acoustic temperature that is output by the CSAT3 sonic is the average of the measurements along all three sonic paths, and therefore the transfer function for sonic temperature becomes

$$H_{T_{sv}}(k_1 p) = \frac{\iint\limits_{-\infty}^{\infty} \left( \sum_{j=1}^{3} \mathrm{sinc}(\mathbf{k} \cdot \mathbf{p}_j/2) \right)^2 \Phi_{T_{sv}}(\mathbf{k})\, dk_2\, dk_3}{\iint\limits_{-\infty}^{\infty} \Phi_{T_{sv}}(\mathbf{k})\, dk_2\, dk_3} \quad . \tag{45}$$

Here $\mathbf{p}_j = \hat{\mathbf{i}}_1 \cos\theta_j \cos\phi + \hat{\mathbf{i}}_2 \cos\theta_j \sin\phi$, where $\theta_j = \theta_1 + 2(j-1)\pi/3$, and $\theta_1$ is the streamwise direction measured with respect to the CSAT3 'a' path, which is in the plane defined by the upper and lower sonic support arms.

Assuming isotropy and $k$ in the inertial subrange,

$$\Phi_{T_{sv}}(k) \propto \frac{N_{T_{sv}} \epsilon^{-1/3} k^{-5/3}}{2\pi k^2} \tag{46}$$

(Tennekes and Lumley, 1972). Here $k$ is the magnitude of $\mathbf{k}$, $N_{T_{sv}}$ is the rate of dissipation of sonic temperature variance, and $\epsilon$ is the rate of turbulent energy dissipation.

The integral in the denominator of the transfer function can be found analytically,

$$\iint\limits_{-\infty}^{\infty} \Phi_{T_{sv}}(\mathbf{k})\, dk_2\, dk_3 = \frac{3}{5} N_{T_{sv}} \epsilon^{-1/3} k_1^{-5/3} \quad , \tag{47}$$

but the integral in the numerator must be computed numerically. Since the integrand is non–negligible over several decades of $k_2$ and $k_3$, it is suggested that the numerical integration be calculated logarithmically over N decades in each quadrant, e.g. $dk_2 = k_2\, d\ln k_2$, $k_0 \le k_2 \le 10^N k_0$. Accuracy to better than 1% is found for $k_0 = k_1 10^{(3/4 - N/2)}$, $N \ge 5$, and S integration steps per decade, e.g. $\Delta \ln k_2 = \ln(10)/S$, $S \ge 5$.

It was shown in Section 6.3 that the sonic pulse sequence delay contaminates the temperature spectrum with the velocity spectra and cross–spectra.

Consequently the velocity spectra and cross-spectra must also be corrected for path averaging. Again assuming isotropy and $k$ in the inertial subrange,

$$\Phi_{ij}(k) \propto \frac{\epsilon^{2/3} k^{-5/3}}{4\pi k^4}(k^2 \delta ij - k_i k_j) \quad . \tag{48}$$

For the power spectrum of the velocity component parallel to one of the CSAT3 measurement paths, $\mathbf{k} \cdot \mathbf{p}_j = k_1 p \cos\alpha + k_3 p \sin\alpha = k_\alpha p$ where $\cos\alpha = \cos\theta_j \cos\phi$, and

$$H_\alpha(k_1 p) = \frac{\iint_{-\infty}^{\infty} \operatorname{sinc}^2(k_\alpha p/2)\,(k^2 - k_\alpha^2)\,dk_2\,dk_3}{\iint_{-\infty}^{\infty} (k^2 - k_\alpha^2)\,dk_2\,dk_3} \quad . \tag{49}$$

Both the integration in the numerator over $k_2$ and the integral in the denominator can be found analytically, so that

$$H_\alpha(k_1 p) = \frac{11\,\Gamma(4/3)\,k_1^{5/3}}{2\sqrt{\pi}\,\Gamma(5/6)\,(4 - \cos^2\alpha)} \int_{-\infty}^{\infty} \operatorname{sinc}^2(k_\alpha p/2)\,(k_1^2 + k_3^2 - k_\alpha^2)\,dk_3 \quad . \tag{50}$$

(Note that this results differs from that of Nielsen and Larsen (2002), Eq. 17.) It is suggested that the integral over $k_3$ again be computed logarithmically, as detailed above.

In the absence of a complete spectral model for the cospectra and quadrature spectra required in Eq. 36, we follow the suggestion of Larsen et al. (1993) and Nielsen and Larsen (2002) to approximate path averaging for the cross spectra as $H_{x,y} = \sqrt{H_x H_y}$, where $x, y$ is any mixed combination of $T_{sv}$, $w_a$, $w_b$, or $w_c$. They justify this approximation with the observation that the cospectra and quadrature spectra terms in Eq. 36 are smaller than the power spectral terms.

## 6.5 Net spectral correction

Finally, the contributions of aliasing, white noise, sonic pulse sequence delays, and sonic path averaging can be combined to obtain an equation for the measured temperature spectrum,

$$\tilde{S}_{T_S}(f) = S_{T_{sv}}(f) \sum_{m=-\infty}^{\infty} H_{T_{sv}}^{p}(k_{1m})\, H_{T_{sv}}^{\tau}(\omega_m)\,(f_m/f)^{-5/3} + 4\sigma_n^2/f_N$$

$$
+ \frac{a^2}{9} S_{w_a} \sum_{m=-\infty}^{\infty} H^p_{w_a}(k_{1m}) \sin^2(\omega_m \tau)\,(f_m/f)^{-5/3} + \ldots
$$

$$
- \frac{a}{9} Co_{w_a T_{sv}} \sum_{m=-\infty}^{\infty} [H^p_{w_a}(k_{1m}) H^p_{Tsv}(k_{1m})]^{1/2} \sin(2\omega_m \tau) \Big(1 + 2\cos(4\omega_m \tau)\Big) \times
$$

$$
\sin(4\omega_m \tau)\,(f_m/f)^{-7/3} + \ldots \quad , \tag{51}
$$

where $k_{1m} = \omega_m / U$, $H^p$ are transfer functions for sonic path averaging, e.g. Eq. 50 for $H^p_{w_j}$, and $H^\tau$ are transfer functions for the sonic pulse sequence delays, e.g. Eq. 37 for $H^\tau_{T_{sv}}$. Note that $S_{w_a}$ in this equation is the true spectrum, which can be found from Eq. 38

$$
S_{w_a}(f) \simeq \tilde{S}_{w_a} \Big/ \sum_{m=-\infty}^{\infty} H^p_{w_a}(k_{1_m}) \cos^2(\omega_m \tau)\,(f_m/f)^{-5/3} \tag{52}
$$

This and similar equations (from Eqs. 39–42) are used to obtain the true spectral variables, which are then substituted into Eq. 51, which is in turn itself solved for $S_{T_{sv}}$. Only three representative terms of Eq. 36 are explicitly written out in Eq. 51; the remainder follow by direct analogy. Note in particular that the velocity–temperature cross spectrum is assumed to decay as $f^{-7/3}$ in the inertial subrange.

# 7 Applicability of Taylor's hypothesis

All of the above has assumed that Taylor's hypothesis can be used to transform the spectra from the frequency to wavenumber domain. Wyngaard and Clifford (1974) show that making this assumption induces an error in the power spectra as a function of turbulence intensity. For a scalar quantity, the ratio of the measured to correct one-dimensional power spectrum is

$$
\frac{S^m}{S} = 1 - \frac{1}{9}\frac{\overline{u'u'}}{\overline{U}^2} + \frac{1}{3}\frac{\overline{v'v'}}{\overline{U}^2} + \frac{1}{3}\frac{\overline{w'w'}}{\overline{U}^2}. \tag{53}
$$

They evaluated this correction for the moderately unstable surface-layer and mixed-layer in their Table 2 and show that this correction is 1–5%.

# 8 Summary

The use of sonic anemometers to determine $C_n^2$ from $T_s$ is straightforward:

1. The measured power spectrum $S_{T_S}$ is obtained from the time series.
2. An estimate of sonic white noise is subtracted from the measured power spectrum.
3. The measured spectra and cross–spectra for sonic path-parallel velocities and the velocity-temperature cross spectra are divided by the appropriate transfer function to correct for aliasing, sonic path averaging, and pulse sequence delays, e.g. Eq. 52.
4. Estimates (from step 3) of cross contamination terms associated with the velocity components parallel to the sonic paths are subtracted from the power spectrum of step 2.
5. $S_{T_s}$ from step 4 is divided by a transfer function to correct for aliasing, sonic path averaging, and pulse sequence delays.
6. The spectral amplitude is estimated in the inertial subrange.
7. Eq. 53 is used to correct for deviations from Taylor's hypothesis.
8. Eq. 3 is used to calculate $C_{T_s}$.
9. Eqs. 26 and 28 are used to compute $C_n^2$ and $C_{na}^2$.

From other data we've collected, we expect the Bowen ratio to be greater than 0.3 at least 85% of the time (98% of clear-sky cases!) which allows the expression $0.03/\beta$ to be ignored in Eq. 26. However, we plan to measure humidity fluctuations as well, if only to measure the humidity flux and thus the Bowen ratio. The Bowen ratio correction in Eq. 26 is derived for $r_{TQ} = \pm 1$, but a similar formulation can be made for other values. With our time series measurements of $T_s$ and $Q$, we can easily determine $r_{TQ}$, albeit at lower frequencies, due to the separation between the humidity and temperature sensors.

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