COMMENT ON ‘FOOTPRINT ANALYSIS: A CLOSED ANALYTICAL SOLUTION BASED ON HEIGHT-DEPENDENT PROFILES OF WIND SPEED AND EDDY VISCOSITY’

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1. Introduction

Haenel and Grünhage (1999; hereafter HG99) recently criticised the Horst (1997, 1999; H97/99) flux footprint model, in particular questioning use of the Gryning et al. (1983; G83) formulation for the exponent r in the crosswind–integrated surface–source diffusion model,

$$\overline{D}^y(x, z) = \frac{A}{\overline{z}\,U}\,\mathrm{e}^{-(z/b\overline{z})^r}. \tag{1}$$

Here $\overline{D}^y$ is the crosswind-integrated vertical concentration distribution within a plume downwind of a unit-strength surface point-source, x and z are the streamwise and vertical coordinates, $\overline{z}$ is the mean plume height, U is the plume advection speed, and A and b are constants dependent on the value of the exponent r as defined in H97/99. HG99 note that the G83 formulation for r causes the footprint model to violate a fundamental constraint on the streamwise integral of the flux footprint and conclude that it should therefore not be used. HG99 subsequently propose a footprint model that is based on a power–law wind profile and that identically satisfies the constraint.

It should be noted that this author was cognizant of the inconsistencies in the H97/99 model highlighted by HG99, and in fact the quantitative consequences were discussed briefly in those papers. However, that issue was considered to be not of sufficient interest to warrant detailed discussion in the literature. It will be shown that the errors caused by the inconsistencies are less than 5% and are therefore negligible compared to other sources of uncertainty in the application of surface-source diffusion and flux-footprint models to the real world, such as the applicability of empirical Monin–Obukhov (MO) surface-flux-layer similarity

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formulae over all but the most homogeneous surfaces or the approximate nature of Equation (1), even over ideal terrain and using optimum values for r. Nevertheless, the opportunity to clarify this issue is welcomed.

This paper discusses, in sequence, the Horst and Weil (1992; HW92) and H97/99 footprint models for direct flux measurements, the proposed HG99 model, and the H97/99 footprint models for gradient and profile flux measurements. It is shown that the formula proposed by HG99 for the exponent r provides results that are significantly less physically realistic than those provided by the G83 formula for r.

2. Horst (1992, 1997/1999) Footprint Models for Direct Flux Measurements

HW92 formally define the flux footprint to be the contribution, per unit emission, of each element of the upwind surface area to a measured vertical flux. The flux footprint is a proportional weighting function, which is consequently required to satisfy the mathematical condition

$$I_\infty \equiv \int_0^\infty \overline{f}^y(x', z)\, dx' = 1, \tag{2}$$

where $\overline{f}^y$ is the crosswind-integrated flux footprint and I_∞ is the cumulative footprint for the case $x \to \infty$. HW92 show that the flux footprint is equal to the spatial distribution of the vertical flux downwind of a unit-strength surface point-source. The constraint of Equation (2) then also follows from the physical requirement of conservation of mass.

HG99 assert that the H97/99 footprint model is deficient because it fails to identically satisfy Equation (2). Although this is a valid observation, HG99 erroneously attribute departures from (2) on the order of 10–15% principally to the fact that the footprint model uses the distance-dependent Gryning et al. (1983) model for r,

$$r = 1 + \frac{c\overline{z}}{\phi_h(c\overline{z}/L)}\left[\frac{\partial \phi_h}{\partial z}\right]_{c\overline{z}} + \frac{\phi_m(c\overline{z}/L)}{\ln(c\overline{z}/z_0) - \psi_m(c\overline{z}/L)} \tag{3}$$

(see Appendix). Here ψ_m, ϕ_m and ϕ_h are the empirical MO-similarity surface-layer functions that describe the dependence on atmospheric stability of the wind profile and the eddy diffusivities for momentum and sensible heat. z_o and L are the micrometeorological roughness and Obukhov lengths, and the parameter c is defined below in Equation (5). However, the failure of the H97/99 model to exactly satisfy Equation (2) is caused mostly by that model's use of several approximations to the exact solution, motivated by a desire to reduce its computational requirements. In addition, it appears that HG99 may also have made an error in the calculation of r from Equation (3).

In order to demonstrate that use of the G83 formula for r satisfies Equation (2) within a few percent, we begin with the fundamental flux-footprint model of HW92. There the flux footprint is calculated by substituting the surface-source vertical diffusion model, Equation (1), into the cross-stream and vertically-integrated advection-diffusion equation,

$$\overline{f}^y(x, z_m) = -\int_0^{z_m} \overline{u}(z)\,\frac{\partial \overline{D}^y(x,z)}{\partial x}\,dz\,. \tag{4}$$

Here z_m is the measurement height and $\overline{u}(z)$ is the mean wind speed profile. The mean plume height $\overline{z}$ is calculated from van Ulden's (1978) model, as discussed in Horst and Weil (1994, 1995), which has a common theoretical basis with the G83 formula for r. The plume advection speed is defined as

$$U(x) \equiv \frac{\int_0^\infty \overline{u}(z)\,\overline{D}^y(x,z)\,dz}{\int_0^\infty \overline{D}^y(x,z)\,dz} \equiv \overline{u}(c\overline{z}). \tag{5}$$

Note the definition here of the parameter c, which is also used in the G83 formula for r (see Appendix A). Since $\overline{D}^y$ depends on r, calculation of the flux footprint requires simultaneous solution of Equations (3) and (5) for r and c. Solution of those equations is simplified by the fact that Equation (5) can be integrated analytically for a log-linear wind profile, that is for neutral and stable stratification,

$$U(x) = (u_*/k)\,[\ln(c_{ln}\,\overline{z}/z_o) + \beta\,\overline{z}/L\,], \tag{6}$$

where c_{ln} pertains to a logarithmic wind profile,

$$c_{ln} = b \exp\left[Ab \int_0^\infty \ln(y)\,\mathrm{e}^{-y^r}\,dy\right]. \tag{7}$$

Here u_* is friction velocity, k is the von Karman constant, and β is the empirical constant in the well-known log-linear wind profile. Equation (5) cannot be integrated analytically for unstable stratification, but to a good approximation,

$$U(x) \simeq (u_*/k)\,[\ln(c_{ln}\,\overline{z}/z_o) - \psi_m(c_{ln}\,\overline{z}/L)\,] = \overline{u}(c_{ln}\overline{z}) \tag{8}$$

(van Ulden, 1978; Horst and Weil, 1994). Note that for stable stratification, $U = \overline{u}(c\overline{z}) \neq \overline{u}(c_{ln}\overline{z})$. HG99 appear to have not recognized this distinction and incorrectly use a formula for c_{ln}, their Equation (8), to solve Equation (3) for r in stable stratification.

Calculation of $\overline{f}^y$ with Equation (4) and a realistic wind profile generally requires numerical integration. In order to further simplify calculation of the flux footprint, Horst and Weil (1994) solved (4) analytically by assuming that $\overline{u}(z)/U(x) = \overline{u}(z)/\overline{u}(c\overline{z})$ is a function solely of $z/\overline{z}$,

$$\overline{f}^y \simeq \frac{d\overline{z}}{dx}\,\frac{z_m}{\overline{z}^2}\,\frac{\overline{u}\,(z_m)}{U(\overline{z})}\,A\,\mathrm{e}^{-(z_m/b\overline{z})^r}\,. \tag{9}$$

TABLE I

I_∞ for fixed (HG99) and distance-dependent (G83) values of r; $z_m/z_o = 40$.

z_m/L	r (HG99)	I_∞ (HW92)		I_∞ (H97/99)	
		r (HG99)	r (G83)	r (HG99)	r (G83)
−0.4	1.41	1.02	1.04	1.08	1.13
−0.04	1.50	1.02	1.05	1.09	1.15
−0.004	1.54	1.01	1.03	1.09	1.11
0	1.54	1.01	1.01	1.09	1.09
0.004	1.55	1.01	1.00	1.08	1.08
0.04	1.62	1.01	0.99	1.05	1.03
0.4	2.05	1.00	1.00	0.98	0.98

Horst (1999) notes that this approximation is exact for a power-law wind profile. Note also that the derivation of Equation (9) is not dependent on the use of MO similarity, as incorrectly suggested by HG99. Equation (9) is sufficiently accurate for practical applications that it is recommended for use in the H97/99 footprint model.

Table I lists values of I_∞ calculated with both the fundamental flux-footprint model, Equation (4), and the approximate footprint model, Equation (9), using in each case both the distance-dependent G83 formula for r and distance-independent values for r. The distance-independent values of r were calculated with the formula proposed by HG99, Equation (13), but I_∞ is insensitive to the specific values. The data in Table I were calculated for a range of atmospheric stability, z_m/L, and for $z_m/z_o = 40$, which appears to be roughly the value used in the HG99 cumulative normalized footprint examples. The third column of Table I lists I_∞ values for the fundamental flux footprint model, using distance-independent values of r. Note the satisfactory agreement with Equation (2), particularly for stable and neutral stratification. The small deviations from unity are of little consequence. The residual deviations are slightly greater for unstable stratification than for stable stratification, which is likely caused by the approximation for U in unstable stratification. Note that Equation (8) is reproduced in HG99 as an equality, their Equation (7), suggesting that they may not have recognized that it is an approximation.

The fourth column of Table I lists I_∞ values for the fundamental footprint model, using the G83 distance-dependent formula for r. By comparing with the adjacent column, it can be seen that the use of Equation (3) introduces additional deviations of 0–3%, depending on stability. In the author's opinion, these are also of little practical consequence.

The last two columns of Table I list I_∞ values for the approximate footprint model of H97/99, again for distance-independent and distance-dependent formulae for r. The last column corresponds to the A-model computations of HG99. Comparison of columns 3 and 4 to columns 5 and 6 shows that the streamwise integral of the approximate footprint model, Equation (9), deviates from unity by an additional 5–10%, which is consistent with the accuracy of this model as discussed in Horst and Weil (1994).

Thus it is apparent that the inability of the H97/99 footprint model (for directly-measured fluxes) to identically satisfy Equation (2) is caused mostly by the approximation for $\overline{f}^y$, rather than by the distance-dependent nature of Equation (3) for r. Note that H97/99 made an ad hoc adjustment for the failure to exactly satisfy Equation (2) by normalizing $\overline{f}^y$ with the computed value of I_∞. This minor adjustment was not mentioned in H97/99 but, in retrospect, may be of interest to those wishing to apply the model.

3. Haenel and Grünhage (1999) Footprint Model

The surface-source diffusion model, Equation (1) is derived from a solution of the two-dimensional advection-diffusion equation

$$\overline{u}(z)\,\frac{\partial \overline{D}^y}{\partial x} = -\frac{\partial}{\partial z}K(z)\frac{\partial \overline{D}^y}{\partial z}, \tag{10}$$

based on the assumptions of horizontal homogeneity, power-law profiles for wind speed and eddy diffusivity K, and a constant value for r. HG99 fault the G83 formula because r is found to be a function of $\overline{z}$, which in turn depends on streamwise distance. Their objection is that this implies that the wind and eddy diffusivity profiles are then no longer horizontally homogeneous, because the exponents of the power-law profiles m and n are directly related to r through $r = 2 + m - n$ (e.g., van Ulden, 1978). However, the distance dependence of r is caused by fitting the simplistic power-law profiles to realistic MO profiles at the distance-dependent height $z = c\overline{z}$ (see Appendix A). The underlying MO profiles of wind and eddy diffusivity are independent of distance.

Nevertheless, in order to obtain a footprint model that identically satisfies the constraint of Equation (2), HG99 propose using a power-law wind profile,

$$\overline{u}(z) = u_r\,(z/z_r)^m\,, \tag{11}$$

rather than the diabatic wind profile described by MO-similarity functions. Here u_r is the wind speed at the reference height z_r and, from the conjugate power law, $m = (r - 1)/2$ (e.g., Pasquill, 1974). The HG99 model is, in fact, a special case of the approximate footprint model, Equation (9), which H99 notes to be an exact solution for a power-law wind profile. The power-law wind profile footprint is obtained by substituting Equation (11) into (5) to obtain

$$U(x) = \left(\frac{b\bar{z}}{z_r}\right)^{(r-1)/2} \frac{\Gamma\left((r+1)/2r\right)}{\Gamma(1/r)}\, u_r \tag{12}$$

and then substituting (11) and (12) into Equation (9). It is then straightforward to integrate Equation (2) analytically and show that I_∞ is identically one, independent of the value of r.

HG99 also propose an alternate equation to calculate r,

$$r = 1 + 2m = 1 + 2\left(\frac{\phi_m(z_m/L)}{\ln(z_m/z_o) - \psi_m(z_m/L) + \psi_m(z_o/L)}\right). \tag{13}$$

This proposal is based on an ad hoc equation in Haenel and Siebers (1995), which those authors asserted, with no direct validation, to be a 'reasonable way to calculate m'. Note that r in Equation (13) depends on height, which is just as theoretically inconsistent with the power-law solution of Equation (10) as is a dependence on x. The advection-diffusion equation contains derivatives of $\overline{D}^y$ with respect to both x and z, which will introduce additional terms into the equation if r depends on either of those variables. A height-dependent r would also seem to be conceptually inconsistent with its role in specifying the shape of the vertical concentration profile.

Figure 1 compares three footprint models: the fundamental flux-footprint model of HW92 using the G83 formula for r, the H97/99 approximate flux-footprint model using the G83 formula for r, and the HG99 power-law footprint model. Results are shown for $z_m/z_o = 300$ and three different atmospheric stabilities, $z_m/L = -0.3,\ 0,\ 0.3$. Note that Figures 1 and 2 use a larger value for z_m/z_o than that in Table I. The flow field at a height of $z_m/z_o = 40$, as used by HG99, is likely to still be within the influence of the roughness sublayer, and thus MO similarity may not be completely valid at that height (Garratt, 1980).

The greatest differences between the HG99 constant-r model and the two G83-based footprint models are found for unstable stratification, where the greatest differences also occur between the formulae for r of Equations (3) and (13). With the exception of the unstable case, the agreement among the models appears to be acceptable for most applications, but it should be cautioned that we have not tested the HG99 model predictions for other values of z_m/z_o or z_m/L.

4. Horst (1997, 1999) Footprint Models for Profile and Gradient Flux Measurements

HG99 correctly note that I_∞ for the H97/99 footprint model for concentration-profile flux measurements also deviates from unity; this is true as well for the

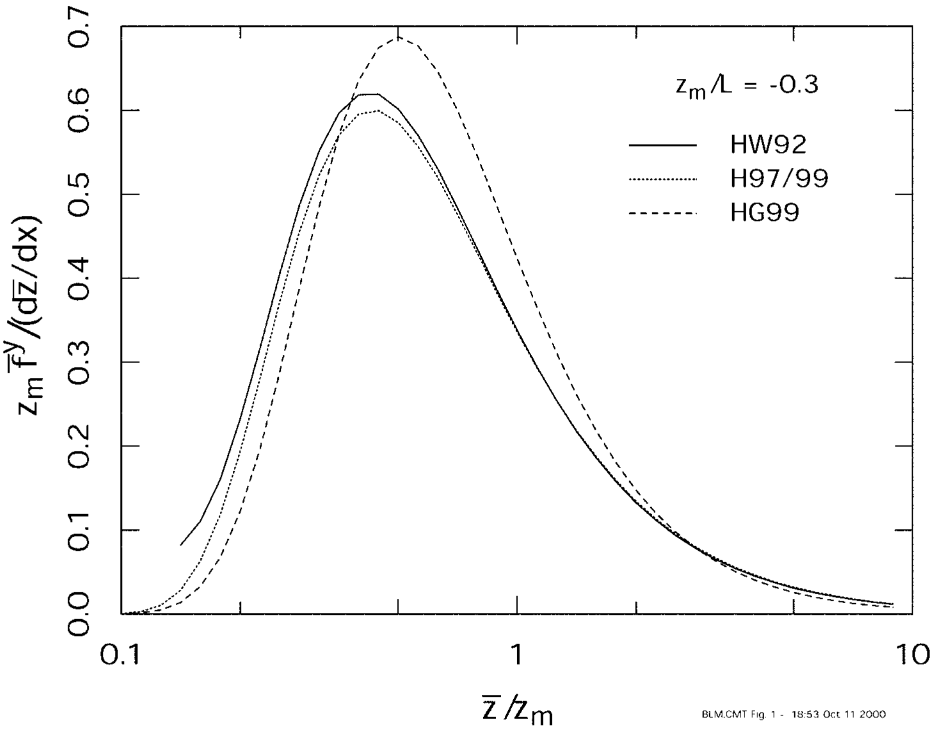

Figure 1. Direct flux footprints as a function of $\bar{z}/z_m$ for $z_m/z_o = 300$ and (a, above) $z_m/L = -0.3$, (b) $z_m/L = 0$, (c) $z_m/L = 0.3$.

H97/99 model for concentration-gradient flux measurements. The deviations from unity are on the order of 5–15%, compared to a few percent for the fundamental flux-footprint model for direct flux measurements. HG99 interpret this to be additional evidence for the lack of validity of the G83 formula for r, because the profile and gradient footprint models contain no approximations other than those contained in the surface-source diffusion model, Equation (1), and the associated formulae for r.

However, as discussed by H97/99, the profile and gradient flux-footprint models are more sensitive to inconsistencies in the surface-source diffusion model because they relate the flux directly to the concentration profile, whereas the footprint model for direct flux measurements relates the flux to the vertical integral of the concentration profile. In addition, H97/99 note that the direct and gradient footprints should be identical because the flux-gradient relation is a property of the underlying flow field. Since a physically-based model for r should apply equally to both the direct and gradient footprint models, a comparison of the two provides a sensitive test of the validity of the surface-source diffusion model, and thus the model for r.

Figure 2 compares the concentration-gradient footprint model using the HG99 formula for r with both the gradient and direct flux footprint models using the G83

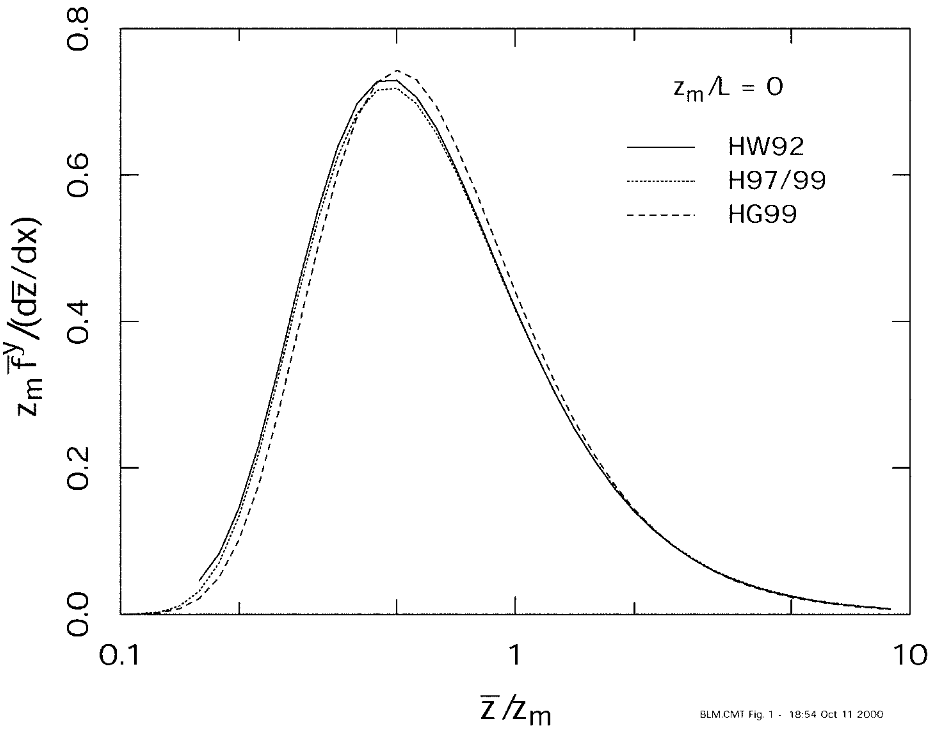

Figure 1b.

formula for r. For the sake of clarity, we have not included the HG99 direct flux footprint in Figure 2. The differences between the concentration-gradient footprint model results and the direct footprint model results are a measure of the inconsistencies of the diffusion models with respect to the advection-diffusion equation, (10). Note that the differences between the concentration-gradient footprint and the direct flux footprint are greater for the HG99 formula for r than for the G83 formula for r. In addition, the values for I_∞ (shown in parenthesis in the figure legends) for the HG99 formula have the greatest departures from unity. Because of the large departures from unity of the HG99 formula, the calculated values for $\overline{f}^y$ have not been normalized by the respective values of I_∞. However, when that is done, $\overline{f}^y$ for the direct flux footprint and for the gradient footprint with the G83 formula for r approach each other (see also Figure 2 of H99), while the gradient footprint with the HG99 formula for r departs even more from the direct flux footprint. This is striking evidence that the G83 formula, Equation (3), is more physically realistic than the HG99 formula, Equation (13). One might argue that the HG99 concentration-gradient footprint should be compared to the HG99 direct flux footprint rather than the fundamental footprint using the G83 formula for r. However, as seen in Figure 1, the differences between the two direct flux footprints

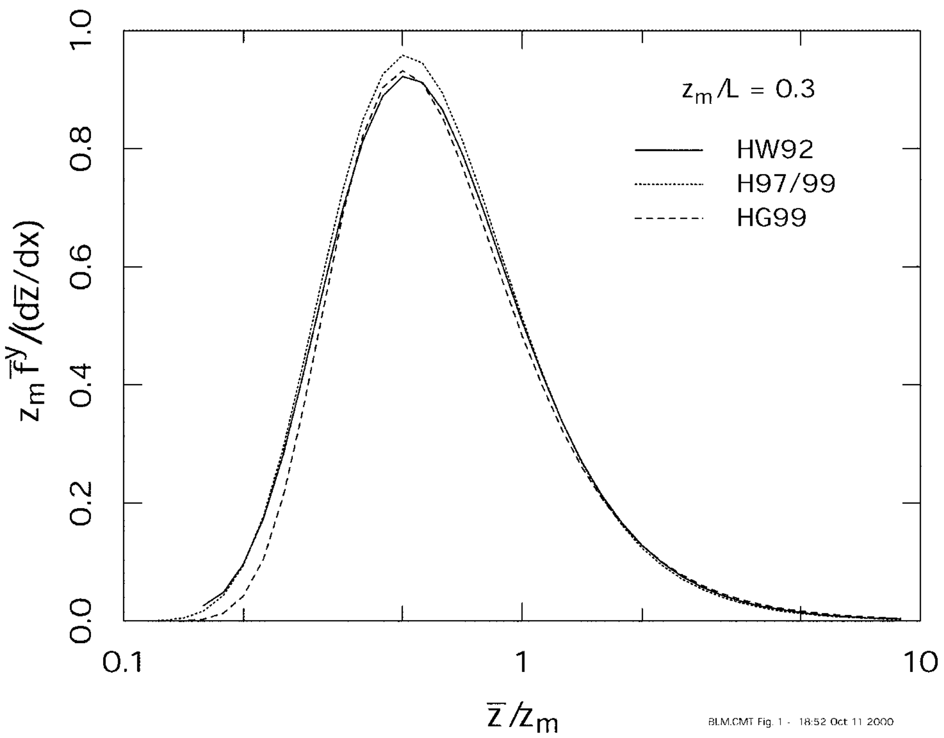

Figure 1c.

are minor compared to the differences between the HG99 gradient footprint and either direct flux footprint, particularly after normalization by I_∞.

5. Summary and Discussion

HG99 have questioned the validity of the distance-dependent G83 formula for r on the basis of the inability of the H97/99 flux footprint model to satisfy the mathematical constraint of Equation (2). Consequently they propose an alternative flux-footprint model that uses an explicit power-law wind profile rather than the MO wind profile formulae, and that uses the Haenel and Siebers (1995) height-dependent formula for the exponent m of the power-law wind profile.

We have shown that use of the G83 formula for r causes deviations from the constraint of Equation (2) that are less than 5%. We consider these to be of little practical consequence. The approximate footprint model recommended by H97/99 causes additional deviations on the order of 5–10%, which we consider to be acceptable for real world applications, particularly when balanced against the decreased computational complexity of this model. HG99 mistakenly attributed the latter errors to use of the G83 formula for r.

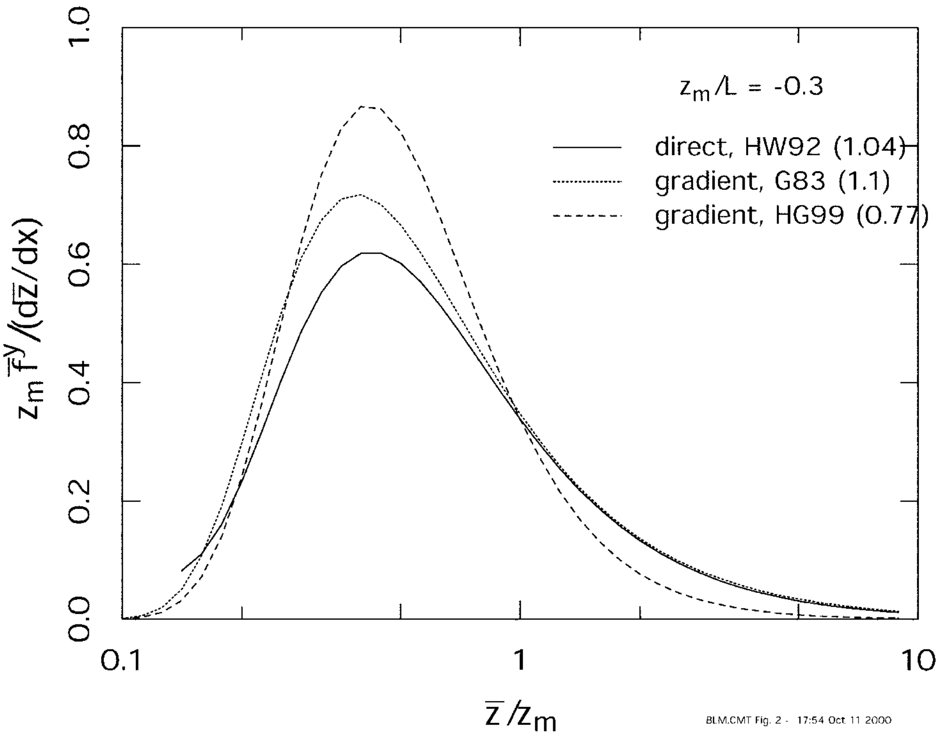

Figure 2a. Direct and concentration-gradient flux footprints as in Figure 1. Values for I_∞ in parentheses.

HG99 also note deviations from Equation (2) on the order of 5–15% for the H97/99 profile flux-footprint model. As discussed in H97/99, these deviations are associated with the failure of the underlying surface-source diffusion model, Equation (1), to identically satisfy the advection-diffusion equation (10). Nevertheless, we are quite pleased with this level of performance of the model. H97/99 demonstrated the accuracy of Equation (1) and the G83 formula for r with a plot of the normalized profile function ϕ_h as calculated with the model, compared to the MO surface-layer similarity formula used in the model equations for $\bar{z}$ and r. An equivalent test is a comparison of the footprints for gradient and direct flux measurements. We have shown here that the performance of the HG99 formula for r is significantly poorer with respect to the latter test than the G83 formula.

We have shown that there are computational errors in HG99, as well as a failure to properly evaluate and comprehend the H97/99 flux-footprint models. However, there also appears to be a fundamental difference between the philosophical approach of HG99 and that of H97/99 and preceeding papers. HG99 have emphasized adherence to the mathematical constraint of Equation (2) at the expense of physical realism, by the use of a power-law wind profile rather than the well-known MO formulae for the wind profile within the surface-flux layer. In contrast, the approach

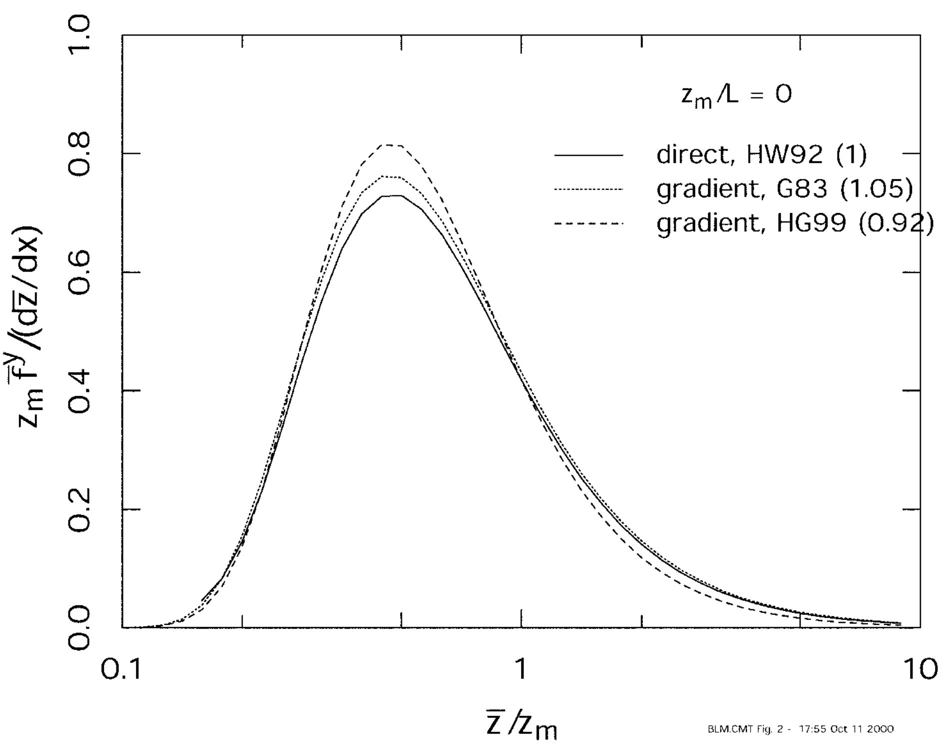

Figure 2b.

of H97/99 and preceding papers has been that the fundamental equation of the flux footprint model is the vertically-integrated advection-diffusion equation (4), and that Equation (1) and the related formulae for $\overline{z}$, U and r are, in the end, justified not so much by their theoretical roots as by their agreement with observations (e.g., van Ulden, 1978; Horst, 1979; Gryning et al., 1983; Finn et al., 1996). Where necessary, compliance with Equation (2) is achieved with the ad hoc procedure of normalization of the flux footprint by its streamwise integral. As shown, these corrections are on the order of 5–15% at most.

HG99 have also proposed a new formula for r with little justification, other than its basis in 'a reasonable way to calculate m', and are undeterred by the fact that the G83 formula provides a noticeably better fit to the data of Finn et al. (1996). In contrast, the G83 formula for r is based directly on numerical solutions of the advection-diffusion equation (see Appendix A), which directly show that r is a function of streamwise distance.

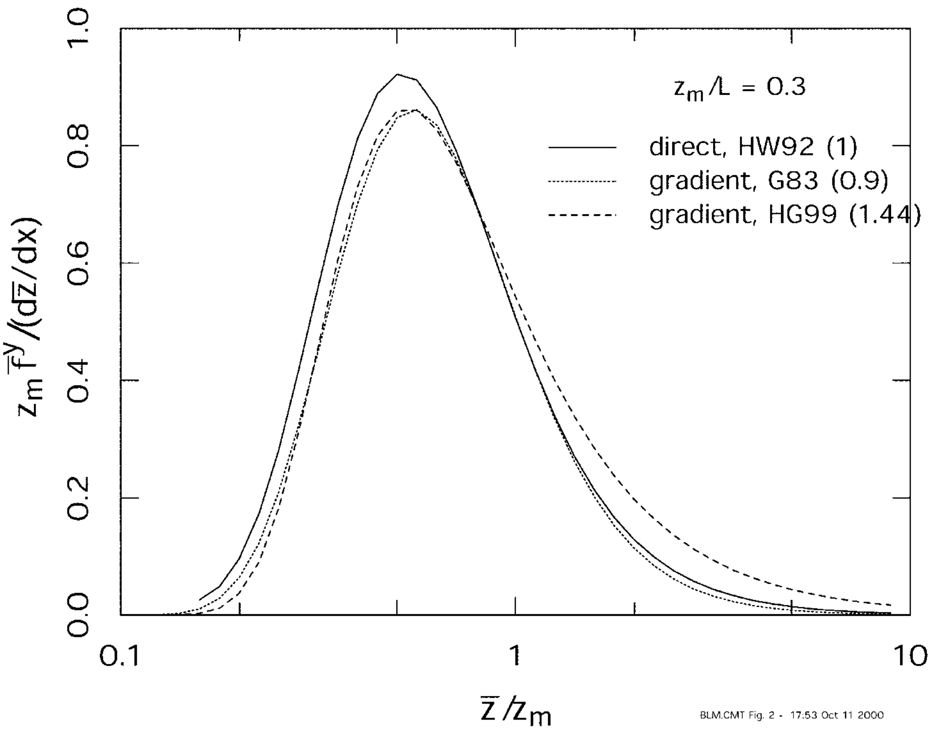

Figure 2c.

Appendix A. The Gryning et al. (1983) Formula for *r*

A derivation of the G83 equation for r begins with the power-law formula

$$r = 2 + m - n = 2 + \left[\frac{\partial \ln \overline{u}(z)}{\partial \ln z}\right]_{z_1} - \left[\frac{\partial \ln K(z)}{\partial \ln z}\right]_{z_1} . \tag{A1}$$

Equation (3) then follows from the definitions of the standard MO relations for the normalized wind and eddy-diffusivity profiles, using $z_1 = c\overline{z}$.

Neither the power-law solution nor MO similarity theory provides a basis for specification of the height z_1. This was provided by G83 on the basis of numerical solutions of the advection-diffusion equation for a surface point-source, using a first-order turbulence closure and MO similarity profiles of wind speed and eddy diffusivity. Values of r were determined from the numerical solutions by fitting the vertical diffusion model, Equation (1), to the numerical concentration profiles. A good match, as a function of streamwise distance and atmospheric stability, between the numerically-determined values of r and Equation (A1) was achieved empirically with $z_1 = c\overline{z}$.

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